Recall: Integration by Parts (IBP)

IBP

$$\int u\,dv = uv - \int v\,du$$

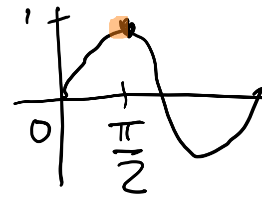
How to decide which is *u* and which is *dv*? **L I A T E**

$u \leftarrow \quad dv \rightarrow$

ex. $\int_0^1 \sin^{-1} x\,dx$ ($\int u\,dv$)

$u = \sin^{-1} x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$



$y = \sin x$

$\sin\frac{\pi}{2} = 1$

$= uv - \int v\,du$

$= x\sin^{-1}x \Big|_0^1 - \int_0^1 x \cdot \frac{1}{\sqrt{1-x^2}}dx$ ← use u-sub

$F(b) - F(a)$

$u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$u_a = 1$

$u_b = 0$

$= \sin^{-1}1 - 0 - \left(-\frac{1}{2}\right)\int_1^0 u^{-1/2}\,du$

$= \frac{\pi}{2} + \frac{1}{\cancel{2}}\,\cancel{2}\,u^{1/2}\Big|_1^0$

$= \frac{\pi}{2} + (\sqrt{0} - \sqrt{1}) = \boxed{\frac{\pi}{2} - 1}$

ex. $\int \frac{dx}{1+4x^2} = \int \frac{1}{1+4x^2}dx$

$(\arctan u)' = \frac{1}{1+u^2}$

$= \int \frac{1}{1+(2x)^2}dx \qquad u = 2x$

$du = 2dx \Rightarrow \frac{1}{2}du = dx$

$= \frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\arctan u + C = \boxed{\frac{1}{2}\arctan(2x) + C}$

Compare and contrast: $\int \frac{dx}{4+x^2} = \int \frac{1}{4+x^2}dx$

↑ want this to be 1

$(\arcsin u)' = \frac{1}{\sqrt{1-u^2}}$

$= \int \frac{1}{4\cdot 1 + \frac{4}{4}x^2}dx$ ← factor out common 4 from denom.

$= \int \frac{1}{4(1+\frac{1}{4}x^2)}dx$

$\frac{1}{4}x^2 = \frac{1}{2}x \cdot \frac{1}{2}x$

$= \frac{1}{4}\int \frac{1}{1+(\frac{1}{2}x)^2}dx$

$u = \frac{1}{2}x \Rightarrow 2du = dx$

$= \frac{1}{4}\cdot 2\int \frac{1}{1+u^2}du = \boxed{\frac{1}{2}\arctan\left(\frac{1}{2}x\right) + C}$

ex. $\int \frac{5x-4}{2x^2+x-1}\,dx$ Why won't u-sub work in this case?

TRIAL: $u = 2x^2 + x - 1$
$du = 4x + 1$ not helpful :(

recall: $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ $\frac{1}{3}\cdot\frac{5}{5} + \frac{2}{5}\cdot\frac{3}{3}$

$\frac{5}{15} + \frac{6}{15} = \frac{11}{15}$ ...

recall: rational functions are $\frac{p(x)}{q(x)}$ $q(x) \neq 0$
p, q are polynomials

Foundation Topic: Adding Rational Functions

$$\frac{1}{x+3}\cdot\frac{x-5}{x-5} + \frac{1}{x-5}\cdot\frac{x+3}{x+3} = \frac{x-5+x+3}{(x+3)(x-5)} = \frac{2x-2}{x^2-2x-15} \Leftrightarrow \frac{1}{x+3} + \frac{1}{x-5}$$

To express rational function as a sum of two fractions, put constants as place markers:

$$\frac{2x-2}{x^2-2x-15} \to \frac{2x-2}{(x+3)(x-5)} \to \frac{\cancel{A}\,1}{x+3} + \frac{\cancel{B}\,1}{x-5}$$

to solve for A, B use LCD to elim denom.

$$\frac{2x-2}{\cancel{(x+3)(x-5)}}\cancel{(x+3)(x-5)} = \frac{A}{\cancel{x+3}}\cancel{(x+3)}(x-5) + \frac{B}{\cancel{x-5}}(x+3)\cancel{(x-5)}$$

LCD

$$2x-2 = A(x-5) + B(x+3)$$

$x=5$: $8 = +8B \Rightarrow B = 1$

$x=-3$: $\frac{-8}{-8} = \frac{\cancel{-8}A}{-8} \Rightarrow A = 1$

New Integration Tool: Partial Fractions

Sometimes, to integrate a rational function, it must be expressed as the sum of two simpler functions.

ex. $\int \frac{2x-2}{(x+3)(x-5)}\,dx \xrightarrow{\text{rewrite}} \int\left(\frac{1}{x+3} + \frac{1}{x-5}\right)dx$

$$= \int \frac{1}{x+3}\,dx + \int \frac{1}{x-5}\,dx$$

$$= \boxed{\ln|x+3| + \ln|x-5| + C}$$

$(\ln u)' = \frac{1}{u}$

$\therefore \int \frac{1}{u}\,du = \ln|u| + C$

ex. $\int \frac{7x-23}{x^2-7x+10}dx = \int \frac{7x-23}{(x-5)(x-2)}dx = \int \left(\frac{A}{x-5}+\frac{B}{x-2}\right)dx$

$= \int \frac{A}{x-5}dx + \int \frac{B}{x-2}dx$

$= A\int \frac{1}{x-5}dx + B\int \frac{1}{x-2}dx$

$= 4\int \frac{1}{x-5}dx + 3\int \frac{1}{x-2}dx$

$= \boxed{4\ln|x-5| + 3\ln|x-2| + C}$

find A and B:

$\frac{7x-23}{\cancel{(x-5)}\cancel{(x-2)}}\cancel{(x-5)}\cancel{(x-2)} = \frac{A}{\cancel{x-5}}\cancel{(x-5)}(x-2) + \frac{B}{\cancel{x-2}}(x-5)\cancel{(x-2)}$

$7x-23 = A(x-2) + B(x-5)$

$x=2 \quad 14-23 = -3B$

$-9 = -3B \Rightarrow B = 3$

$x=5 \quad 12 = 3A \Rightarrow A = 4$

Do: $\int \frac{9x-13}{x^2+x-12}dx = \int \frac{9x-13}{(x+4)(x-3)}dx = A\int \frac{1}{x+4}dx + B\int \frac{1}{x-3}dx$

$= 7\int \frac{1}{x+4}dx + 2\int \frac{1}{x-3}dx$

$= \boxed{7\ln|x+4| + 2\ln|x-3| + C}$

$\frac{9x-13}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$

$9x-13 = A(x-3) + B(x+4)$

$14 = 7B \Rightarrow B = 2$

$-36-13 = -7A$

$-49 = -7A \Rightarrow A = 7$

ax^2+bx+c

Do: factor denominator $2x^2+x-1 = (\ x+1\ )(\ 2x-1\ )$ $AC = 2\cdot 1$

$= 2x^2+2x-x-1$

$= 2x(x+1)-1(x+1)$

$= (x+1)(2x-1)$

Do: $\int e^{7x}dx$

$= \boxed{\frac{1}{7}e^{7x}+C}$

ex. $\int \frac{1}{2x+1}dx$ $u=2x+1$, $\frac{1}{2}du = dx$

$= \frac{1}{2}\int \frac{1}{u}du$

$= \boxed{\frac{1}{2}\ln|2x+1|+C}$

now, let's integrate: $\int \frac{5x-4}{2x^2+x-1}dx$

$= \int \frac{5x-4}{(x+1)(2x-1)}dx = A\int \frac{1}{x+1}dx + B\int \frac{1}{2x-1}dx$

$= 3\int \frac{1}{x+1}dx - 1\int \frac{1}{2x-1}dx$

$= \boxed{3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C}$

$\frac{5x-4}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$

$5x-4 = A(2x-1)+B(x+1)$

$x=-1$: $-5-4 = A(-3)$

$-9 = -3A \Rightarrow A = 3$

$x=\frac{1}{2}$: $\frac{5}{2}-4\frac{2}{2} = B\left(\frac{1}{2}+1\right)$

$-\frac{3}{2} = \frac{3}{2}B \Rightarrow B=-1$

What happens when there are more than two factors in denominator?

ex. $\int \frac{x-5}{2x^3+7x^2-4x}dx = \int \frac{x-5}{x(2x^2+7x-4)}dx$

$= \int \frac{x-5}{x(x+4)(2x-1)}dx$

$= A\int \frac{1}{x}dx + B\int \frac{1}{x+4}dx + C\int \frac{1}{2x-1}dx$

$AC = 8 \cdot 1$

$2x^2+7x-4$

$= 2x^2+8x - x - 4$

$= 2x(x+4) - 1(x+4)$

$= (x+4)(2x-1)$

try to find on your own $\Rightarrow A = \frac{5}{4} \quad B = -\frac{1}{4} \quad C = -2$